18.100A PROBLEM SET 2 SOLUTIONS

Problem 1. *Let* $a_n = \frac{1}{\ln n}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)$ *for* $n \geq 2$. *Show* $\lim_{n\to\infty} a_n = 1$.

Proof. Since $\frac{1}{x}$ is a decreasing function for $x > 0$, we have

$$\int_1^{n+1} \frac{1}{x} dx \leq 1 + \frac{1}{2} + \cdots + \frac{1}{n} \leq 1 + \int_1^n \frac{1}{x} dx.$$

In addition, direct computations yield

$$\int_1^{n+1} \frac{1}{x} dx = \ln(n+1) \geq \ln n, \qquad 1 + \int_1^n \frac{1}{x} dx = 1 + \ln n.$$

Therefore,

$$1 = \frac{\ln n}{\ln n} \leq a_n \leq \frac{1 + \ln n}{\ln n} = 1 + \frac{1}{\ln n}.$$

Given $\epsilon > 0$, for $n > e^{1/\epsilon}$ the following holds

$$|a_n - 1| \leq \frac{1}{\ln n} \leq \frac{1}{\ln e^{1/\epsilon}} = \epsilon.$$

Therefore, 1 is the limit. □

Comments : The limit multiplication theorem is not available for $b_n = 1 + \cdots + \frac{1}{n}$ and $c_n = \frac{1}{\ln n}$, because b_n does not have a limit and tends to $+\infty$.

Problem 2. *Let* $a_n \geq 0$ *and* $\lim_{n\to\infty} a_n = L$. *Prove that* $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{L}$.

Proof. In the case $L = 0$, given $\epsilon > 0$ we have $|a_n| \leq \epsilon^2$ for $n \gg 1$. Therefore, $|\sqrt{a_n}| \leq \epsilon$ for $n \gg 1$, namely $\lim \sqrt{a_n} = 0$.

We now assume $L > 0$. Then,

$$|\sqrt{a_n} - \sqrt{L}| \leq \left|\frac{a_n - L}{\sqrt{a_n} + \sqrt{L}}\right| \leq \frac{|a_n - L|}{\sqrt{L}}. \tag{1}$$

Since $\lim a_n = L$, given $\epsilon > 0$ we have $|a_n - L| \leq \sqrt{L}\epsilon$ for $n \gg 1$. Thus, $|\sqrt{a_n} - \sqrt{L}| \leq \epsilon$ for $n \gg 1$, namely $\lim \sqrt{a_n} = \sqrt{L}$.

□

Comments :

(1) In the case $L = 0$, the last term in the inequality (1) is not defined.
(2) There is no theorem in the textbook which guarantees the convergence of $\sqrt{a_n}$. Hence, one can not assume that $\sqrt{a_n}$ converges to a certain limit M.

Problem 3. *Problem 5-2 page 75. (It is enough to give one proof of (b), while the textbook asks to find two proofs.)*

Proof for (a). Since $1 - L > 0$, we have $\frac{a_{n+1}}{a_n} \le L + (1 - L) = 1$ for $n \gg 1$, namely $a_{n+1} \le a_n$ for $n \gg 1$. □

Comments : If $L = 1$ then $L - 1 = 0$ and thus the above argument fails. For example, $a_n = 1 - \frac{1}{n}$ is increasing. However, $\lim \frac{a_{n+1}}{a_n} = 1$.

Proof for (b). Since $\frac{1-L}{2} > 0$, we have $\frac{a_{n+1}}{a_n} \le L + \frac{1-L}{2} = \frac{1+L}{2}$ for $n \ge N$ where N is a large natural number. Let M denote $\frac{1+L}{2}$. Then,

$$a_{N+1} \le M a_N.$$

We assume $a_{N+k} \le M^k a_N$ for a natural number k. Then,

$$a_{N+k+1} = \frac{a_{N+k+1}}{a_{N+k}} a_{N+k} \le M \cdot M^k a_N = M^{k+1} a_N.$$

By the mathematical induction, we have $a_n \le M^{n-N} a_N$ for $n \ge N$.

Theorem 3.4 shows $\lim M^n = 0$. Therefore, Theorem 5.1.1 implies $\lim M^{n-N} a_N = 0$. Since $0 < a_n$, the squeeze theorem yields $\lim a_n = 0$.

□

Problem 4. *Problem 5-7 page 75. (Hint: consider the two cases (1) $a_0 \ge 2$, and (2) $0 < a_0 < 2$.)*

Proof for (a). Suppose $a_0 \ge 2$. Then, $a_1 = \sqrt{2a_0} \ge 2$. Next, we assume $a_k \ge 2$ for some k. Then, $a_{k+1} = \sqrt{2a_k} \ge 2$. Therefore, by the mathematical induction, we have $a_n \ge 2$ for all n.

Since $a_n \ge 2$ implies $a_{n+1} = \sqrt{2a_n} \le a_n$, a_n is decreasing and bounded below. Hence, by the completeness property, a_n converges.

Suppose $0 < a_0 < 2$. Then, $a_1 = \sqrt{2a_0} < 2$. Next, we assume $a_k < 2$ for some k. Then, $a_{k+1} = \sqrt{2a_k} < 2$. Therefore, by the mathematical induction, we have $a_n < 2$ for all n.

Since $a_n < 2$ implies $a_{n+1} = \sqrt{2a_n} > a_n$, a_n is increasing and bounded above. Hence, by the completeness property, a_n converges. □

Proof for (b). Let L be the limit $\lim a_n = L$. Then, a_{n+1} also has the limit L. Thus, Theorem 5.1 implies

$$L = \lim a_{n+1} = \lim 2a_n^2 = 2 \lim a_n^2 = 2(\lim a_n)^2 = 2L^2.$$

Hence, $L = 0$ or 2. However, if $a_0 \ge 2$ then $a_n \ge 2$ by the above proof of (a). So, the limit location theorem implies $L \ge 2$, namely $L \ne 0$. If $0 < a_0 < 2$ then $a_n \ge a_0$ by the above proof of (a). So, the limit location theorem implies $L \ge a_0 > 0$, namely $L \ne 0$. In conclusion, $L \ne 0$ and thus $L = 2$. □

Comments : Even if $a_n > 0$, the limit can be zero as like $a_n = \frac{1}{n}$. Thus, one show use $a_n > a_0$ and $a_0 > 0$.

Problem 5. *Exercise 6.3.1. Page 90.*

Proof. In (a) and (c), we have $0 \le b_n \le 1$. Hence, the BW theorem guarantees that there exists a convergent subsequence.

Regarding (b), if $a_n = -1 + \frac{1}{n}$ then $b_n = -n + 1$ which is divergent. □

Problem 6. *Exercise 6.4.1. Page 90.*

Proof. Let $\lim a_n = L$. Then, given $\epsilon > 0$ we have $|a_n - L| < \frac{\epsilon}{2}$ for $n \ge N$ where N is a large integer. Therefore,

$$|a_n - a_m| = |(a_n - L) + (L - a_m)| \le |a_n - L| + |L - a_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

for $n, m \ge N$. Namely, a_n is a Cauchy sequence. □

Comments : $|a_n - a_m| < \epsilon$ should hold for any $n, m \ge N$, namely n and m are independent. Some student only showed $|a_n - a_{n+1}| < \epsilon$ and it is not enough to be a Cauchy sequence.

Problem 7. *Exercise 6.5.1. (b), (d) Page 90.*

Proof for (b). If $n = 2m$ then $a_{2m} = \frac{1}{2m}$. Hence, $0 < a_{2m} \le \frac{1}{2}$ and $a_2 = \frac{1}{2}$.

If $n = 2m-1$ then $a_{2m-1} = \frac{-1}{2m-1}$. Hence, $-1 \le a_{2m-1} < 0$ and $a_1 = -1$.

Therefore, $\frac{1}{2}$ is the maximum and -1 is the minimum. Property 6.5A and 6.5B show that $\frac{1}{2}$ is the supremum and -1 is the infimum. □

Proof for (d). We have

$$a_n - a_{n+1} = \frac{n}{2^n} - \frac{n+1}{2^{n+1}} = \frac{2n - (n+1)}{2^{n+1}} = \frac{n-1}{2^{n+1}} \ge 0.$$

Hence, a_n is decreasing, namely $a_n \le a_1 = \frac{1}{2}$. Therefore, $\frac{1}{2}$ is the maximum. Property 6.5A shows that $\frac{1}{2}$ is the supremum.

We have $a_n > 0$ and

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{2n} = \frac{1}{2} + \frac{1}{2n}.$$

Thus, Theorem 5.1 shows $\lim a_{n+1}/a_n = 1/2 < 1$. Hence, the above proof for the problem 3-(b) shows $\lim a_n = 0$.

$0 < a_n$ implies that 0 is a lower bounds. Assume that there exists a lower bound $b > 0$. However, $\lim a_n = 0$ implies $a_n \le b/2 < b$ for $n \gg 1$. Thus, there is no lower bound greater than 0. Hence, 0 is the infimum.

Assume that the minimum exists. Then, it is 0 by Property 6.5B. However, $a_n \ne 0$. Hence, the minimum does not exists. □

Problem 8. *Exercise 6.5.4. Page 90.*

We provide two different proofs.

Proof 1. By the completeness property $\sup S$ and $\inf T$ exists.

Given $t \in T$, $s \leq t$ holds for all $s \in S$. Hence, every t is an upper bound for S. Since $\sup S$ is the least upper bound, we have $\sup S \leq t$ for all $t \in T$. Hence, $\sup S$ is a lower bound for T. Since $\inf T$ is the greatest lower bound, we have $\sup S \leq \inf T$. □

Proof 2. By the completeness property $\sup S$ and $\inf T$ exists.

Suppose $\sup S > \inf T$. Then, the number $m = (\sup S + \inf T)/2$ satisfies $\sup S > m > \inf T$. Then, there exists an element $s_0 \in S$ such that $s_0 > m$. If not, $s \leq m$ holds for all $s \in S$, namely m is an upper bound. But it is impossible since the least upper bound $\sup S$ is greater than m. In the same manner there exists an element $t_0 \in T$ such that $t_0 < m$. Then, we have $s_0 > m > t_0$ which contradicts to the condition $s < t$. □

Problem 9. *Problem 6-2 Page 91.*

Proof for (a). For each natural number n, there exists an element $a_n \in S$ such that

$$\bar{m} - \frac{1}{n} < a_n \leq \bar{m}. \tag{2}$$

(Remark that $a_n = a_m$ is possible for $n \neq m$.) If not, $s \leq \bar{m} - \frac{1}{n}$ holds for all $s \in S$, namely $\bar{m} - \frac{1}{n}$ is an upper bound. But it is impossible since the least upper bound $\sup S = \bar{m}$ is greater than $\bar{m} - \frac{1}{n}$.

Then, given $\epsilon > 0$, for $n > 1/\epsilon$ we have

$$|a_n - \bar{m}| \leq \frac{1}{m} < \epsilon,$$

namely a_n converges to $\bar{m}$. □

Comments :

(1) For the cases $S = \{1\}$ or $S = [0,1] \cup \{2\}$, one should include the equality for the right inequality in 2.

(2) If S includes any interval with positive length, S has uncountably many elements. Namely, one can not give orders for the all elements in S.

Proof for (b). By the completeness property $\sup A$, $\sup B$, and $\sup A + B$ exists.

Given $a \in A$ and $b \in B$, we have $a + b \leq \sup A + \sup B$, namely $\sup A + \sup B$ is an upper bound for $A + B$. Therefore, we have

$$\sup A + B \leq \sup A + \sup B. \tag{3}$$

On the other hand, for all $a \in A$ and $b \in B$ we have

$$a + b \leq \sup A + B.$$

Hence, give $a \in A$ the following holds

$$b \leq (\sup A + B) - a,$$

for all $b \in B$. Thus,

$$\sup B \leq (\sup A + B) - a.$$

Hence,

$$a \leq (\sup A + B) - \sup B,$$

hold for all $a \in A$. Therefore,

$$\sup A \leq (\sup A + B) - \sup B,$$

namely $\sup A + \sup B \leq \sup A + B$. Thus, combining with 3 yields the desired result. □

Problem 10. *Problem 6-3 Page 91.*

Proof for (a). Since f is decreasing, we have

$$f(n+1) \leq \int_n^{n+1} f(x)dx \leq f(n),$$

namely

$$0 \leq \int_n^{n+1} f(x)dx - f(n+1) \leq f(n) - f(n+1). \tag{4}$$

Hence, for $m > n$ we have

$$a_m - a_n = \sum_n^{m-1} f(k) - \int_n^m f(x)dx = \sum_n^{m-1} f(k) - \int_k^{k+1} f(x)dx.$$

Combining the above equality with 4 and $f \geq 0$ yields

$$0 \leq a_m - a_n \leq \sum_n^{m-1} f(k) - f(k+1) = f(n) - f(m) \leq f(n).$$

Given ϵ, there exists a large integer N such that $f(n) < \epsilon$ for $n \geq N$. Since $a_m - a_n = 0$ for $m = n$, we have

$$|a_m - a_n| \leq f(n) < \epsilon,$$

for $m, n \geq N$. □

Proof for (b).

$$a_n = 1 + \cdots + e^{n-1} - \int_0^n e^{-x}dx = \frac{1 - e^{-n}}{1 - e^{-1}} + e^{-x}\Big|_0^n = \frac{1 - e^{-n}}{1 - e^{-1}} + e^{-n} - 1.$$

We have $\lim e^{-n} = 0$ by Theorem 3.4. So, Theorem 5.1 implies

$$\lim a_n = \frac{1}{1 - e^{-1}} - 1 = \frac{e^{-1}}{1 - e^{-1}} = \frac{1}{e - 1}.$$

□